Square Root of $\left[\begin{smallmatrix}\mathbf{T}_{\boldsymbol{q}} & \mathbf{0}\\ \mathbf{0} & 1\end{smallmatrix}\right]$

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Let $\boldsymbol{q}$ be a unit quaternion. Its inverse is its conjugate $\boldsymbol{q}^*$. Recall the quaternion operators $\otimes$ and $\circledast$ and the $[\boldsymbol{q}\otimes]$ and $[\boldsymbol{q}\circledast]$ operator matrices, which were defined in [1, App.] and elsewhere. For quaternions $\boldsymbol{p}$ and $\boldsymbol{r}$, we have $\boldsymbol{r} = [\boldsymbol{q}\otimes]\boldsymbol{p} = \boldsymbol{q}\otimes\boldsymbol{p}$ and $\boldsymbol{r} = [\boldsymbol{p}\circledast]\boldsymbol{q} = \boldsymbol{p}\circledast\boldsymbol{q}$.

Define the matrix

$$C = \begin{bmatrix} -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \tag{1}$$

Note that $C^2 = I$. We could call C a conjugation operator matrix since $\boldsymbol{q}^* = C\boldsymbol{q}$. It is easily verified that

$$C[\boldsymbol{q}\otimes]C = [\boldsymbol{q}^*\circledast] = [\boldsymbol{q}\circledast]^T \tag{2}$$

from which we have that

$$C[\boldsymbol{q}\otimes] = [\boldsymbol{q}\circledast]^T C \tag{3a}$$

and

$$[\boldsymbol{q}\otimes]C = C[\boldsymbol{q}\circledast]^T \tag{3b}$$

It is well known that

$$[\boldsymbol{q}\circledast]^T[\boldsymbol{q}\otimes] = [\boldsymbol{q}\otimes][\boldsymbol{q}\circledast]^T = \begin{bmatrix} \mathbf{T}_{\boldsymbol{q}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \tag{4}$$

where $\mathbf{T}_{\boldsymbol{q}}$ is the direction cosine or attitude matrix that corresponds to the quaternion $\boldsymbol{q}$.

From the foregoing relations, we conclude that $C[\boldsymbol{q}\otimes]$ and $[\boldsymbol{q}\otimes]C$ are two square roots of $\left[\begin{smallmatrix}\mathbf{T}_{\boldsymbol{q}} & \mathbf{0}\\ \mathbf{0} & 1\end{smallmatrix}\right]$ because, from (3a) and (4), we have

$$\begin{aligned} \big(C[\boldsymbol{q}\otimes]\big)^2 &= \big(C[\boldsymbol{q}\otimes]\big)\big(C[\boldsymbol{q}\otimes]\big) \\ &= C[\boldsymbol{q}\otimes]C[\boldsymbol{q}\otimes] \\ &= [\boldsymbol{q}\circledast]^T[\boldsymbol{q}\otimes] \\ &= \begin{bmatrix} \mathbf{T}_{\boldsymbol{q}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned} \tag{5a}$$

and similarly from (3b) and (4)

$$\begin{aligned} \big([\boldsymbol{q}\otimes]C\big)^2 &= \big([\boldsymbol{q}\otimes]C\big)\big([\boldsymbol{q}\otimes]C\big) \\ &= [\boldsymbol{q}\otimes]C[\boldsymbol{q}\otimes]C \\ &= [\boldsymbol{q}\otimes][\boldsymbol{q}\circledast]^T \\ &= \begin{bmatrix} \mathbf{T}_{\boldsymbol{q}} & \mathbf{0} \\ \mathbf{0} & 1 \end{bmatrix} \end{aligned} \tag{5b}$$

Define

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{e} \\ c \end{bmatrix} \qquad [\boldsymbol{q}\otimes] = \begin{bmatrix} c\mathbf{I} - [\boldsymbol{e}\times] & \boldsymbol{e} \\ -\boldsymbol{e}^T & c \end{bmatrix} \tag{6}$$

where $\boldsymbol{e}$ is the vector part of the quaternion and c is the scalar part. Then from the (5b), we have

$$\mathbf{T}_{\boldsymbol{q}} = \begin{bmatrix} -c\mathbf{I} + [\boldsymbol{e}\times] & \boldsymbol{e} \end{bmatrix} \begin{bmatrix} -c\mathbf{I} + [\boldsymbol{e}\times] \\ \boldsymbol{e}^T \end{bmatrix} \tag{7}$$

If we start from (5a), the result differs only in a sign that cancels out. We have, then,

$$\mathbf{T}_{\boldsymbol{q}} = \left(c\mathbf{I} - [\boldsymbol{e}\times]\right)^2 + \boldsymbol{e}\boldsymbol{e}^T \tag{8a}$$

$$= \mathbf{I} - 2c[\boldsymbol{e}\times] + 2[\boldsymbol{e}\times]^2 \tag{8b}$$

Equation (8b) is a common expression for $\mathbf{T}_{\boldsymbol{q}}$ in terms of $\boldsymbol{q}$, whereas (8a) is a (new?) quadratic form. The quadratic form could be used to transform a vector via the following steps

$$\begin{aligned} \boldsymbol{v}' &= c\boldsymbol{v} - \boldsymbol{e} \times \boldsymbol{v} \\ \boldsymbol{v}'' &= c\boldsymbol{v}' - \boldsymbol{e} \times \boldsymbol{v}' + \boldsymbol{e}(\boldsymbol{e}^T\boldsymbol{v}) \end{aligned}$$

so that $\boldsymbol{v}'' = \mathbf{T}_{\boldsymbol{q}}\boldsymbol{v}$.

Observe that a factorization of $\mathbf{T}_{\boldsymbol{q}}$ can be obtained from a QR decomposition, viz

$$\begin{bmatrix} -c\mathbf{I} + [\boldsymbol{e}\times] \\ \boldsymbol{e}^T \end{bmatrix} = QR = [Q_1 \ Q_2] \begin{bmatrix} R_1 \\ \mathbf{0} \end{bmatrix} = Q_1 R_1 \tag{9}$$

where Q is orthogonal and R_1 is upper triangular. A factorization of $\mathbf{T}_{\boldsymbol{q}}$ is then

$$\mathbf{T}_{\boldsymbol{q}} = R_1^T R_1 \tag{10}$$

Note that the process of computing R_1 is essentially one of completing the square in equation (8a).

Acknowledgement

Landis Markley pointed out that the matrix square root was derived by Bernard Friedland [2].

References

[1] Pittelkau, M. E., "Measurement Sensitivity Equations in Attitude Determination", *2003 Flight Mechanics Symposium*, NASA Goddard Space Flight Center, NASA CP-2003-212246, 28–30 Oct. 2003.

[2] Bernard Friedland, "Analysis Strapdown Navigation Using Quaternions," IEEE AES-14, No. 5, 1978, pp. 764–768.